Admixture of quasi-Dirac and Majorana neutrinos with tri-bimaximal mixing

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We propose a realization of the so-called *bimodal/schizophrenic* model proposed recently. We assume S_4, the permutation group of four objects as flavor symmetry giving tri-bimaximal lepton mixing at leading order. In these models the second massive neutrino state is assumed quasi-Dirac and the remaining neutrinos are Majorana states. In the case of inverse mass hierarchy, the lower bound on the neutrinoless double beta decay parameter m_{ee} is about two times that of the usual lower bound, within the range of sensitivity of the next generation of experiments.

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I. INTRODUCTION

Charged particles are Dirac fermions while electrically neutral fermions, like neutrinos, can be either Dirac or Majorana. Neutrinoless double beta decay $0\nu\beta\beta$ experiments will confirm (if observed) the Majorana nature of neutrinos [1]. Experiments for $0\nu\beta\beta$ currently under construction will have sensitivity in the range of the inverse hierarchy mass spectrum [2–5]. Recently, it has been observed in [6] that if the second massive neutrino is of Dirac type (and so does not participate to the $0\nu\beta\beta$ decay) in the case of inverse mass hierarchy, the lower bound on the $0\nu\beta\beta$ parameter m_{ee} is about two times that of the usual bound. In reference [6], they forbid the Majorana mass for the second neutrino at tree level by means of a flavor symmetry.

The parameter m_{ee} can be written as combination of neutrino masses, namely $m_{ee} = \sum_{i=1}^{3} U_{ei}^2 m_{\nu_i}$ where U is the lepton mixing matrix. In the inverse hierarchy case, when three neutrinos are of Majorana type, we have

$$|m_{ee}| \approx |(\cos^2\theta_{12} + e^{i\alpha}\sin^2\theta_{12})m_{\rm atm}| > \frac{m_{\rm atm}}{3} \approx 17\,\text{meV}. \tag{1}$$

If the second massive neutrino is of Dirac type, that is $m_{\nu_2} = 0$ in m_{ee} we have

$$|m_{ee}| \approx |\cos^2\theta_{12} m_{\rm atm}| > \frac{2m_{\rm atm}}{3} \approx 34\,\text{meV}. \tag{2}$$

Such a value is in the range of sensitivity of the next generation of experiments and could be ruled out very soon.

A four component spinor ψ is a Majorana spinor if $\psi = \psi^c$ where ψ^c is the charge conjugate of ψ. The Dirac mass term for a massive spin 1/2 fermion is given by

$$-m\bar{\psi}\psi \tag{3}$$

where $\psi = (\chi, \sigma_2\phi^*)$ and χ, ϕ are two component spinors. Assuming $\chi = \frac{1}{\sqrt{2}}(\rho_2 + i\rho_1)$, $\phi = \frac{1}{\sqrt{2}}(\rho_2 - i\rho_1)$, a four component Dirac mass term (3) is equivalent to two Majorana mass terms of equal mass and opposite parity[7, 8]

$$-m\bar{\psi}\psi = -\frac{m}{2}(\rho_1^T\sigma_2\rho_1 + \rho_2^T\sigma_2\rho_2). \tag{4}$$

For an arbitrary number of Majorana neutrinos the neutrino mass matrix is given by

$$\mathcal{L} = -\frac{1}{2}\sum_{i,j}^{n} M_{ij}\rho_i^T\sigma_2\rho_j, \tag{5}$$

In general the eigenvalues of the mass matrix M can have different signs and we can assign a signature matrix diag$(+,+,...,-,-,..)$. For two neutrino states we can have diag$(+,-)$ or diag$(+,+)$. In the former case, if the

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absolute value of the masses is the same, the two neutrino types make up a Dirac neutrino. When the two neutrinos are active-sterile we have the so-called quasi-Dirac neutrino [9] and when they are active-active we have the so called pseudo-Dirac neutrino [10].

In Ref. [6] the second massive neutrino state has a quasi-Dirac mass [1], while the first and third neutrinos get a Majorana mass a la seesaw. Since each flavor state is an admixture of quasi-Dirac and Majorana states, they call such a case *schizophrenic*. For recent studies on this subject see also [11–13]. There are several models in the literature for exact tri-bimaximal [14] based on the group of permutation of four objects S_4 as flavor symmetry [15–29]. Here we study the schizophrenic case assuming the S_4 group with extra abelian symmetries as flavor symmetry. Breaking S_4 into different Z_2 subgroups respectively in the charged lepton and neutrino sectors we obtain tri-bimaximal mixing at tree-level. The difference between our model and the model of Ref. [6] is that they assume the permutation of three objects S_3 flavor symmetry instead of S_4 and they obtain tri-bimaximal mixing only assuming the charged lepton mass matrix to be diagonal, while in our model the charged lepton mass matrix is diagonal at tree-level by means of S_4.

The Letter is organized as follow: in section II we present the model, in section III we give the neutrino and charged lepton mass matrices, in section IV we study the problem of the vacuum alignments and we give our conclusions.

II. THE MODEL

We extend the Standard Model (SM) with a $G_f = S_4 \times Z_3 \times Z_3' \times Z_3''$ flavor symmetry where S_4 is the permutation group of four objects, Z_3, Z_3', Z_3'' are abelian groups characterized respectively by $\omega^3 = 1$, ${\omega'}^3 = 1$ and ${\omega''}^3 = 1$. In order to simplify the study of the S_4-alignments of the scalar fields we assume supersymmetry, therefore all the fields are assumed to be superfields. We also add three right-handed neutrinos and eight scalar isosinglets called flavons. We assume ν_2^c to be a singlet of S_4 and ν_1^c, ν_3^c to form a doublet ν_D^c of S_4. The $SU_L(2)$ doublet L and singlet l^c are both triplets 3_1 of S_4. The matter content of the model is given in table I.

	L	l^c	ν_2^c	ν_D^c	$h^{u,d}$	ϕ_ν	ξ_ν	φ_l	χ_l	$\tilde{\chi}_l$	φ_ν	σ	$\tilde{\sigma}$
S_4	3_1	3_1	1_1	2	1_1	3_1	1_1	2	1_1	1_1	2	1_1	1_1
Z_3	1	ω^2	1	1	1	1	1	ω	ω	ω^2	1	1	1
Z_3'	${\omega'}^2$	ω'	ω'	1	1	ω'	${\omega'}^2$	1	1	1	1	1	1
Z_3''	1	1	1	ω''	1	1	1	1	1	1	ω''	${\omega''}^2$	ω''

TABLE I: Matter content of the model.

The relevant Yukawa terms of the superpotential invariant under G_f are

$$\begin{aligned} w_l &= \frac{y_{1l}}{M_\Lambda} L l^c h^d \chi_l + \frac{y_{2l}}{M_\Lambda} L l^c h^d \varphi_l, \\ w_\nu &= \frac{y_{2\nu}}{M_\Lambda^2} L \nu_2^c h^u \phi_\nu \xi_\nu + \frac{y_{1\nu}}{M_\Lambda^2} L \nu_D^c h^u \phi_\nu \sigma + y_\sigma \nu_D^c \nu_D^c \tilde{\sigma} + y_\varphi \nu_D^c \nu_D^c \varphi_\nu . \end{aligned} \tag{6}$$

Since ν_2^c is charged under Z_3' the mass term $\nu_2^c \nu_2^c$ is forbidden. The scalar flavons take vacuum expectation value (vev) along the following direction of S_4 (see section IV)

$$\langle \phi_\nu \rangle \sim (1,1,1), \quad \langle \varphi_\nu \rangle \sim (0,1), \quad \langle \varphi_l \rangle \sim (-\sqrt{3},1). \tag{7}$$

When the scalar flavons take such vevs, the elements ST^2, S^2TS, TS, S^3T^2 leave invariant the charged leptons while the elements TST, $TSTS^2$, S, S^3 leave invariant the neutrino sector. Here S and T are generators of S_4, see the Appendix A. The different breaking in the charged lepton and neutrino sectors gives (at tree-level) tri-bimaximal mixing. The scalar S_4 singlets ξ_ν, χ_l, $\tilde{\chi}_l$, σ and $\tilde{\sigma}$ take vevs different from zero.

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[1] At leading order m_{ν_2} is a Dirac state, but at next to leading order it takes a small Majorana mass resulting in a quasi-Dirac state.

III. MASS MATRICES

From the superpotential w_ν and the vevs alignments given in eq. (7) the Dirac couplings for the neutrinos are proportional to the following S_4 contractions

$$(L\phi)_{1_1}\nu_2^c \sim (L_e + L_\mu + L_\tau)\,\nu_2^c \tag{8}$$

$$(L\phi)_2\nu_D^c \sim \begin{pmatrix} \frac{1}{\sqrt{2}}(L_\mu - L_\tau) \\ \frac{1}{\sqrt{6}}(-2L_e + L_\mu + L_\tau) \end{pmatrix} \times \begin{pmatrix} \nu_1^c \\ \nu_3^c \end{pmatrix}. \tag{9}$$

Then the Dirac neutrino mass matrix is given by

$$m_D = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} m^D_{\nu_1} & 0 & 0 \\ 0 & m^D_{\nu_2} & 0 \\ 0 & 0 & m^D_{\nu_3} \end{pmatrix}, \tag{10}$$

where

$$m^D_{\nu_2} = \frac{y_{2\nu}}{M_\Lambda^2}\langle h^u\rangle\langle\phi_\nu\rangle\langle\xi_\nu\rangle, \qquad m^D_{\nu_1} = m^D_{\nu_3} = \frac{y_{1\nu}}{M_\Lambda^2}\langle h^u\rangle\langle\phi_\nu\rangle\langle\sigma\rangle. \tag{11}$$

The right-handed Majorana neutrino mass matrix is given by

$$M_R = \begin{pmatrix} y_\sigma\langle\tilde\sigma\rangle + y_\varphi\langle\varphi_\nu\rangle & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_\sigma\langle\tilde\sigma\rangle - y_\varphi\langle\varphi_\nu\rangle \end{pmatrix} \equiv \begin{pmatrix} M_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M_3 \end{pmatrix}. \tag{12}$$

where $M_1 \neq M_3$. The neutrino mass matrix is diagonalized by the tri-bimaximal mixing matrix, see eq. (10). One neutrino has a quasi-Dirac mass[2] $m_{\nu_2} \equiv m^D_{\nu_2}$ (see eq. (11)) and two neutrinos have Majorana masses

$$m_{\nu_1} = -\frac{{m^D_{\nu_1}}^2}{M_1}, \quad m_{\nu_3} = -\frac{{m^D_{\nu_1}}^2}{M_3}. \tag{13}$$

Note that the masses m_{ν_3} and m_{ν_1} are proportional one to each other, so the atmospheric mass spliting arises from the M_1 and M_3 mass splitting.

Assuming Yukawa couplings of order one and the following value for the scales where the scalar fields take vev

$$\begin{array}{lccccccc} & \langle h^{u,d}\rangle & < & \langle\xi_\nu\rangle \sim \langle\tilde\sigma\rangle \sim \langle\varphi_\nu\rangle & < & \langle\sigma\rangle \sim \langle\phi_\nu\rangle \sim \langle\chi_l\rangle \sim \langle\varphi_l\rangle \sim \langle\tilde\chi_l\rangle & < & M_\Lambda \\ \text{scales (GeV)}: & 10^2 & , & 10^5 & , & 10^{13} & , & 10^{15} \end{array} \tag{14}$$

then the neutrino masses m_{ν_1}, m_{ν_2} and m_{ν_3} are at the eV scale with $M_R \sim 10^5\,GeV$. As a particular example, taking

$$y_{1\nu} = 0.2200, \quad y_{2\nu} = 0.6345, \quad y_\varphi = 1, \quad y_\sigma = -0.2300, \tag{15}$$

we have

$$|m_{\nu_1}| = 0.0628\,eV, \quad |m^D_{\nu_2}| = 0.0634\,eV, \quad |m_{\nu_3}| = 0.0393\,eV, \tag{16}$$

giving about $\Delta m^2_{\rm sol} \approx 7.5\cdot 10^{-5} eV^2$ and $\Delta m^2_{\rm atm} \approx 2.4\cdot 10^{-3} eV^2$ in agreement with data. We observe that the next to leading order term $\nu_2^c\nu_2^c\xi_\nu^2/M_\Lambda$ is allowed giving a contribution to M_R of order 10^{-5} GeV that is negligible.

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[2] Next to leading order terms as well as loop corrections generate a negligible mass term for ν_2^c then we have a quasi-Dirac state instead of a Dirac one.

The charged lepton mass matrix is given from the superpotential w_l. It is not difficult to show that the resulting mass matrix is diagonal. This arises from the S_4 symmetry and the masses are given as[3]

$$\begin{aligned} m_e &= \frac{y_{1l}}{M_\Lambda}\langle h^d\rangle\langle\chi_l\rangle - \frac{2y_{2l}}{\sqrt{6}M_\Lambda}\langle h^d\rangle\langle\varphi_{l2}\rangle\,, & (17)\\ m_\mu &= \frac{y_{1l}}{M_\Lambda}\langle h^d\rangle\langle\chi_l\rangle + \frac{y_{2l}}{M_\Lambda}\langle h^d\rangle(\frac{1}{\sqrt{6}}\langle\varphi_{l2}\rangle + \frac{1}{\sqrt{2}}\langle\varphi_{l1}\rangle), & (18)\\ m_\tau &= \frac{y_{1l}}{M_\Lambda}\langle h^d\rangle\langle\chi_l\rangle + \frac{y_{2l}}{M_\Lambda}\langle h^d\rangle(\frac{1}{\sqrt{6}}\langle\varphi_{l2}\rangle - \frac{1}{\sqrt{2}}\langle\varphi_{l1}\rangle). & (19)\end{aligned}$$

If $\langle\varphi_{l1}\rangle$ and $\langle\varphi_{l2}\rangle$ are free, we have three combinations of free parameters and we can fit the charged lepton masses as given below

$$\begin{aligned} \frac{y_{1l}}{M_\Lambda}\langle h^d\rangle\langle\chi_l\rangle &= \frac{m_e+m_\mu+m_\tau}{3}, & (20)\\ \frac{y_{2l}}{M_\Lambda}\langle h^d\rangle\langle\varphi_{l1}\rangle &= \frac{m_\mu-m_\tau}{\sqrt{2}}, & (21)\\ \frac{y_{2l}}{M_\Lambda}\langle h^d\rangle\langle\varphi_{l2}\rangle &= \frac{-2m_e+m_\mu+m_\tau}{\sqrt{6}}, & (22)\end{aligned}$$

that are of order of the mass of the τ, in agreement with the assumption in eq. (14). In the limit $m_{e,\mu}\to 0$ from eqs. (21) and (22) we have

$$\frac{\langle\varphi_{l1}\rangle}{\langle\varphi_{l2}\rangle} = -\sqrt{3}, \tag{23}$$

in agreement with the vev alignment given in eq. (7). The mass of the muon m_μ arises from a small deviation the alignment $\langle\varphi_l\rangle \sim (-\sqrt{3}(1+\epsilon), 1)$. Such a deviation can arise from next to leading order terms in the scalar superpotential as well as by assuming S_4 soft breaking terms in the superpotential. While the electron mass m_e arises by means of a fine-tuning of the coupling y_{1l}. We can easily accommodate the three charged lepton masses in our model, in particular $m_\mu \ll m_\tau$ arises from the alignment $\langle\varphi_l\rangle \sim (-\sqrt{3}, 1)$.

IV. VACUUM ALIGNMENTS

In the previous sections we showed that assuming the alignments in eq. (7) we obtain tri-bimaximal neutrino mixing and diagonal charged lepton mass matrix. Here we show that the alignment of the flavon fields can arise from the minimization of the superpotential.

The superpotential invariant under $S_4\times Z_3\times Z_3'\times Z_3''$ for the flavon fields of table (I) is given by

$$\begin{aligned} w &= \lambda_1\varphi_l\varphi_l\varphi_l + \lambda_2\varphi_l\varphi_l\chi_l + \lambda_3\chi_l\chi_l\chi_l + \lambda_4\chi_l\tilde{\chi}_l + \lambda_5\tilde{\chi}_l\tilde{\chi}_l\tilde{\chi}_l + \\ &+ \lambda_6\varphi_\nu\varphi_\nu\varphi_\nu + \lambda_7\varphi_\nu\varphi_\nu\tilde{\sigma} + \lambda_8\sigma\tilde{\sigma} + \lambda_9\tilde{\sigma}\tilde{\sigma}\tilde{\sigma} + \lambda_{10}\sigma\sigma\sigma + \\ &+ \lambda_{11}\phi_\nu\phi_\nu\phi_\nu + \lambda_{12}\xi_\nu\xi_\nu\xi_\nu + \mu_\phi\phi_\nu\phi_\nu + \mu_\xi\xi_\nu\xi_\nu\,, \qquad (24)\end{aligned}$$

where the terms proportional to μ_ϕ and μ_ξ break softly the auxiliary Z_3' symmetry while the Z_3 and Z_3'' are preserved in the superpotential. We denote the vevs of the flavon fields as below

$$\begin{gathered} \langle\varphi_l\rangle = (u_1,u_2), \quad \langle\varphi_\nu\rangle = (v_1,v_2), \quad \langle\phi_\nu\rangle = (r_1,r_2,r_3), \\ \langle\chi_l\rangle = v_\chi, \quad \langle\tilde{\chi}_l\rangle = \tilde{v}_\chi, \quad \langle\sigma\rangle = v_\sigma, \quad \langle\tilde{\sigma}\rangle = \tilde{v}_\sigma, \quad \langle\xi_\nu\rangle = v_\xi. \end{gathered} \tag{25}$$

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[3] It is very easy to see that corrections of second order arise by couplings with the flavon $\tilde{\chi}_l$ but those can be reabsorbed in the y_{1l} coupling.

We show below that $r_1 = r_2 = r_3 = r$, $v_1 = 0$, $v_2 = v$, $u_1 = -\sqrt{3}u$ and $u_2 = u$ is a possible solution of the minimization of the superpotential. Then we have to solve the set of equations

$$\begin{aligned}
\frac{\partial w}{\partial u_1} &= -\lambda_1 6\sqrt{3}u^2 - \lambda_2 2\sqrt{3}uv_\chi = 0, &(26)\\
\frac{\partial w}{\partial u_2} &= \lambda_1 6u^2 + \lambda_2 2uv_\chi = 0, &(27)\\
\frac{\partial w}{\partial v_\chi} &= \lambda_1 4u^2 + \lambda_3 3v_\chi^2 + \lambda_4 \tilde{v}_\chi = 0, &(28)\\
\frac{\partial w}{\partial \tilde{v}_\chi} &= \lambda_4 v_\chi + \lambda_5 3\tilde{v}_\chi^2 = 0, &(29)\\
\frac{\partial w}{\partial v_1} &= 0, &(30)\\
\frac{\partial w}{\partial v_2} &= -\lambda_6 3v^2 + \lambda_7 2v\tilde{v}_\sigma = 0, &(31)\\
\frac{\partial w}{\partial v_\sigma} &= \lambda_{10} 3v_\sigma^2 + \lambda_8 \tilde{v}_\sigma = 0, &(32)\\
\frac{\partial w}{\partial \tilde{v}_\sigma} &= \lambda_7 v^2 + \lambda_8 \tilde{v}_\sigma + \lambda_9 3\tilde{v}_\sigma^2 = 0, &(33)\\
\frac{\partial w}{\partial r_1} &= \lambda_{11} r^2 + \mu_\phi 2r = 0, &(34)\\
\frac{\partial w}{\partial r_2} &= \lambda_{11} r^2 + \mu_\phi 2r = 0, &(35)\\
\frac{\partial w}{\partial r_3} &= \lambda_{11} r^2 + \mu_\phi 2r = 0, &(36)\\
\frac{\partial w}{\partial v_\xi} &= \lambda_{12} 3{v_\xi}^2 + \mu_\xi 2v_\xi = 0, &(37)
\end{aligned}$$

where we have assumed $r_1 = r_2 = r_3 = r$, $v_1 = 0$, $v_2 = v$, $u_1 = -\sqrt{3}u$ and $u_2 = u$. It is easy to show that such a system admits a solution with r, v and u different from zero and fixed by the coupling constants of the superpotential in eq. (24).

In summary, we present a realization of the so-called bimodal/schizophrenic ansatz, that is one of the massive neutrino state is of Dirac-type and the remaining two are Majorana. Then each flavor state is an admixture of Dirac and Majorana states giving distinct predictions for the neutrinoless double beta decay rate. The model consist of a supersymmetric extension of the SM based on the $S_4 \times Z_3^3$ flavor symmetry, where we add three right-handed neutrinos, the second of them transforming as a singlet of S_4 and the other two as a doublet of S_4, and eight scalar singlets of the SM. The model also gives tri-bimaximal mixing for neutrinos at leading order. As was pointed out in [6] this kind of models can be ruled out very soon by neutrinoless double beta decay experiments.

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Appendix A: The group S_4

The discrete group S_4 is given by the permutations of four objects and it is composed by 24 elements. It can be defined by two generators S and T that satisfy

$$S^4 = T^3 = 1, \quad ST^2S = T\,. \tag{A1}$$

The 24 elements of S_4 belong to five classes

$$\begin{aligned}
\mathcal{C}_1 &: I\,;\\
\mathcal{C}_2 &: S^2, TS^2T^2, S^2TS^2T^2\,;\\
\mathcal{C}_3 &: T, T^2, S^2T, S^2T^2, STST^2, STS, S^2TS^2, S^3TS\,;\\
\mathcal{C}_4 &: ST^2, T^2S, TST, TSTS^2, STS^2, S^2TS\,;\\
\mathcal{C}_5 &: S, TST^2, ST, TS, S^3, S^3T^2\,.
\end{aligned} \tag{A2}$$

The elements of $\mathcal{C}_{2,4}$ define two different sets of Z_2 subgroups of S_4, the ones of the class $\mathcal{C}_4$ a set of Z_3 abelian discrete symmetries and those belonging to $\mathcal{C}_5$ a set of Z_4 abelian discrete symmetries. The S_4 irreducible representations are two singlets, $1_1, 1_2$, one doublet, 2, and two triplets, 3_1 and 3_2. We adopt the following basis

$$S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad T = -\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\,, \tag{A3}$$

for the doublet representation and

$$S_\pm = \pm\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\,, \tag{A4}$$

for the triplet representations 3_1 and 3_2 respectively. Clearly the generators (S_+, T) and (S_-, T) define the two triplet representations $3_1, 3_2$ respectively. All the product rules can be straightforwardly derived. We remind the reader to the product rules reported in [30] (see also [31]).

The product of S_4 representation:

$$\begin{aligned}
1_i \times 1_j &= 1_{\mathrm{(i+j)mod2+1}} \quad \forall\ \text{i and j},\\
2 \times 1_i &= 2 \quad \forall\ \text{i},\\
3_i \times 1_j &= 3_{\mathrm{(i+j)mod2+1}} \quad \forall\ \text{i and j},
\end{aligned}$$

$$\begin{aligned}
3_i \times 2 &= 3_1 + 3_2 \quad \forall\ \text{i},\\
3_1 \times 3_2 &= 1_2 + 2 + 3_1 + 3_2,
\end{aligned}$$

$$[2\times 2] = 1_1 + 2\,, \quad \{2\times 2\} = 1_2 \quad \text{and} \quad [3_i \times 3_i] = 1_1 + 2 + 3_1\,, \quad \{3_i \times 3_i\} = 3_2 \quad \forall\ \text{i},$$

where we introduced the notation $[\mu \times \mu]$ for the symmetric and $\{\mu \times \mu\}$ for the anti-symmetric part of the product $\mu \times \mu$.

Note that $\nu \times \mu = \mu \times \nu$ for all representations μ and ν. For the irreducible representations:

$$A \sim 1_1 \quad , \quad B \sim 1_2 \quad , \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} \sim 2\,, \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \end{pmatrix} \sim 3_1 \quad \text{and}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \begin{pmatrix} c'_1 \\ c'_2 \\ c'_3 \end{pmatrix} \sim 3_2\,.$$

The explicit products for 1_1 representation with any μ representation:

$$\begin{pmatrix} A\,a_1 \\ A\,a_2 \end{pmatrix} \sim 2 \quad , \quad \begin{pmatrix} A\,b_1 \\ A\,b_2 \\ A\,b_3 \end{pmatrix} \sim 3_1 \quad , \quad \begin{pmatrix} A\,c_1 \\ A\,c_2 \\ A\,c_3 \end{pmatrix} \sim 3_2\,.$$

and the product of 1_2 with the any μ representation:

$$\begin{pmatrix} -B\,a_2 \\ B\,a_1 \end{pmatrix} \sim 2 \quad , \quad \begin{pmatrix} B\,b_1 \\ B\,b_2 \\ B\,b_3 \end{pmatrix} \sim 3_2 \quad , \quad \begin{pmatrix} B\,c_1 \\ B\,c_2 \\ B\,c_3 \end{pmatrix} \sim 3_1 \ .$$

The products of $\mu \times \mu$:

for 2

$$\begin{gathered} a_1 a'_1 + a_2 a'_2 \sim 1_1, \\ -a_1 a'_2 + a_2 a'_1 \sim 1_2, \\ \begin{pmatrix} a_1 a'_2 + a_2 a'_1 \\ a_1 a'_1 - a_2 a'_2 \end{pmatrix} \sim 2, \end{gathered}$$

for 3_1

$$\begin{gathered} \sum_{j=1}^{3} b_j b'_j \sim 1_1, \\ \begin{pmatrix} \frac{1}{\sqrt{2}}(b_2 b'_2 - b_3 b'_3) \\ \frac{1}{\sqrt{6}}(-2b_1 b'_1 + b_2 b'_2 + b_3 b'_3) \end{pmatrix} \sim 2, \\ \begin{pmatrix} b_2 b'_3 + b_3 b'_2 \\ b_1 b'_3 + b_3 b'_1 \\ b_1 b'_2 + b_2 b'_1 \end{pmatrix} \sim 3_1 \ , \quad \begin{pmatrix} b_3 b'_2 - b_2 b'_3 \\ b_1 b'_3 - b_3 b'_1 \\ b_2 b'_1 - b_1 b'_2 \end{pmatrix} \sim 3_2, \end{gathered}$$

for 3_2

$$\begin{gathered} \sum_{j=1}^{3} c_j c'_j \sim 1_1, \\ \begin{pmatrix} \frac{1}{\sqrt{2}}(c_2 c'_2 - c_3 c'_3) \\ \frac{1}{\sqrt{6}}(-2c_1 c'_1 + c_2 c'_2 + c_3 c'_3) \end{pmatrix} \sim 2, \\ \begin{pmatrix} c_2 c'_3 + c_3 c'_2 \\ c_1 c'_3 + c_3 c'_1 \\ c_1 c'_2 + c_2 c'_1 \end{pmatrix} \sim 3_1 \ , \quad \begin{pmatrix} c_3 c'_2 - c_2 c'_3 \\ c_1 c'_3 - c_3 c'_1 \\ c_2 c'_1 - c_1 c'_2 \end{pmatrix} \sim 3_2 \ . \end{gathered}$$

For $2 \times 3_1$:

$$\begin{gathered} \begin{pmatrix} a_2 b_1 \\ -\frac{1}{2}(\sqrt{3} a_1 b_2 + a_2 b_2) \\ \frac{1}{2}(\sqrt{3} a_1 b_3 - a_2 b_3) \end{pmatrix} \sim 3_1 \\ \begin{pmatrix} a_1 b_1 \\ \frac{1}{2}(\sqrt{3} a_2 b_2 - a_1 b_2) \\ -\frac{1}{2}(\sqrt{3} a_2 b_3 + a_1 b_3) \end{pmatrix} \sim 3_2 \end{gathered}$$

and for $2 \times 3_2$

$$\begin{gathered} \begin{pmatrix} a_1 c_1 \\ \frac{1}{2}(\sqrt{3} a_2 c_2 - a_1 c_2) \\ -\frac{1}{2}(\sqrt{3} a_2 c_3 + a_1 c_3) \end{pmatrix} \sim 3_1 \\ \begin{pmatrix} a_2 c_1 \\ -\frac{1}{2}(\sqrt{3} a_1 c_2 + a_2 c_2) \\ \frac{1}{2}(\sqrt{3} a_1 c_3 - a_2 c_3) \end{pmatrix} \sim 3_2. \end{gathered}$$

For $3_1 \times 3_2$

$$\begin{gathered} \sum_{j=1}^{3} b_j c_j \sim 1_2 \\ \begin{pmatrix} \frac{1}{\sqrt{6}}(2b_1 c_1 - b_2 c_2 - b_3 c_3) \\ \frac{1}{\sqrt{2}}(b_2 c_2 - b_3 c_3) \end{pmatrix} \sim 2 \\ \begin{pmatrix} b_3 c_2 - b_2 c_3 \\ b_1 c_3 - b_3 c_1 \\ b_2 c_1 - b_1 c_2 \end{pmatrix} \sim 3_1 \ , \quad \begin{pmatrix} b_2 c_3 + b_3 c_2 \\ b_1 c_3 + b_3 c_1 \\ b_1 c_2 + b_2 c_1 \end{pmatrix} \sim 3_2 \ . \end{gathered}$$

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